

HAMILTONIAN FORMULATION AND EXACT SOLUTIONS OF BIANCHI TYPE-I MODEL IN CONFORMAL GRAVITY

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We develop a Hamiltonian formulation of Bianchi type-I cosmological model in conformal gravity, i.e., the theory described by the Lagrangian $\mathfrak{L} = C_{abcd}C^{abcd}$, which involves the quadratic curvature invariant constructed from the Weyl tensor, in a four-dimensional spacetime. We derive the explicit forms of the super-Hamiltonian and the constraint expressing the conformal invariance of the theory, and we write down the system of canonical equations. To seek out exact solutions to this system we add extra constraints on the canonical variables and we go through a *global involution algorithm* that possibly leads to the closure of the constraint algebra. This enables us to extract all possible particular solutions that may be written in closed analytical form. On the other hand, probing the local analytical structure we show that the system does not possess the Painlevé property (presence of movable logarithms) and that it is therefore not integrable. We stress that there is a very fruitful interplay of *local integrability-related methods* such as the Painlevé test and global techniques such as the involution algorithm. Strictly speaking, we demonstrate that the global involution algorithm has proven to be exhaustive in the search for exact solutions. The conformal relationship of the solutions, or absence thereof, with Einstein spaces is highlighted.

1 Introduction

Unlike what happens in general relativity, where unwanted second-order derivative terms of the metric can be discarded from the gravitational action through pure divergences, there is no chance whatsoever to weed out such terms in the context of higher-order gravity theories. It is fortunate however that a consistent method of building up a Hamiltonian formulation of those theories does actually exist; it is a generalization of the classical *Ostrogradsky formalism*.¹ Basically it consists in introducing auxiliary degrees of freedom that encompass each of the successive derivative terms higher than first order. Without resorting explicitly to this method Boulware worked out a Hamiltonian formulation of quadratic gravity.² Contradistinctively, Buchbinder and Lyakhovich developed a canonical formalism for the most general quadratic gravitational Lagrangian in four dimensions, by employing the aforementioned generalized Ostrogradsky method.³ In a previous work we have already applied Boulware's canonical formalism to Bianchi cosmologies, for the pure R^2 variant of the general quadratic theory.⁴ In this contribution we consider the con-

formally invariant case, which is based on the quadratic Lagrangian density $\mathfrak{L} = \sqrt{-g} C_{abcd} C^{abcd}$, where C^a_{bcd} is the Weyl tensor.^a Making use of a slightly different generalized Ostrograsky construction (as compared to Buchbinder and Lyakhovich's formalism) we derive the canonical form of the conformally invariant action, which was first obtained by Boulware. We further particularize the Hamiltonian formalism to the Bianchi type-I cosmological model and write down the canonical equations. This is our starting point for seeking out exact particular solutions and for asking whether the conformal Bianchi type-I cosmological model is integrable or not. In that respect, we sum up very recent results which can be found elsewhere.⁶

2 Conformal gravity and Bach equations

Consider a four-dimensional spacetime (\mathcal{M}, g) and the quadratic gravitational action

$$S = -\frac{1}{4} \int_{\mathcal{M}} d^4x \sqrt{-g} C_{abcd} C^{abcd}, \quad (1)$$

where C_{abcd} is the Weyl tensor. Its variation with respect to the metric yields the conformally invariant fourth-order equations

$$B_{ab} := 2\nabla^m \nabla^n C_{mabn} + C_{mabn} R^{mn} = 0, \quad (2)$$

which have been put up by Bach⁷ who adopted Weyl's paradigm of a conformally invariant gravitational theory,⁸ though without considering the additional Weyl 1-form. In Eq. (2), B_{ab} is called the Bach tensor; it is symmetric, trace-free and conformally invariant of weight -1 . Although Eq. (2) is a very compact formula it is not appropriate for calculational purpose with computer algebra — the contracted double covariant derivative of the Weyl tensor is rather heavy to compute even for simple metrics. Recently Tsantilis *et al* have provided an algorithm for the MATH-TENSOR package that gives the Bach equations in a much more tractable form, especially with regard to cosmological applications, and which is based on the decomposition of the Riemann tensor in its irreducible pieces.⁹ In accordance with their results we write down the following equivalent expression of B_{ab} , as given by Eq. (2),

$$B_{ab} = -\square \left(R_{ab} - \frac{R}{6} g_{ab} \right) + \frac{1}{3} \nabla_a \nabla_b R + (C_{mabn} + R_{mabn} + R_{mban}) R^{mn}. \quad (3)$$

The simplest cosmological model exhibiting non trivial physical degrees of freedom in the conformally invariant gravitational theory based on the action

^aWe adopt Wald's conventions of sign and definitions of curvature tensors (but the extrinsic curvature).⁵

given in Eq. (1) is the spatially homogeneous anisotropic Bianchi type-I model. (The isotropic FLRW cosmological models are conformally flat.) Writing the metric in such a way that the conformal invariance becomes manifest already from the outset,

$$ds^2 = e^{2\mu} \left[-dt^2 + e^{2(\beta_+ + \sqrt{3}\beta_-)} dx^2 + e^{2(\beta_+ - \sqrt{3}\beta_-)} dy^2 + e^{-4\beta_+} dz^2 \right], \quad (4)$$

the corresponding Bach equations, as given in Eq. (3), can be derived with the help of symbolic computational packages such as the EXCALC package in REDUCE.⁶

3 Hamiltonian Bianchi type-I cosmology

3.1 Hamiltonian formalism and canonical equations

The conformally invariant action in Eq. (1) can be cast into Hamiltonian form by means of a generalized Ostrogradsky construction.

Assume first that spacetime is foliated into a family of Cauchy hypersurfaces Σ_t of unit normal n^a . The induced metric h_{ab} onto these hypersurfaces is defined by the formula $h_{ab} = g_{ab} + n_a n_b$. The way the hypersurfaces are embedded into spacetime is provided by the extrinsic curvature tensor $K_{ab} := -\frac{1}{2} \mathcal{L}_{\vec{n}} h_{ab}$, where $\mathcal{L}_{\vec{n}}$ denotes the Lie derivative along the normal n^a (we adopt $\mathcal{L}_{\vec{n}}$ as a generalized notion of time differentiation). The standard ADM variables are introduced: the *lapse function* N and *shift vector* N^i . The 3 + 1-splitting of spacetime enables us to express the Lagrangian density $\mathfrak{L} = -\frac{1}{4} \sqrt{-g} C_{abcd} C^{abcd}$ only in terms of the quantities defined onto Σ_t . After some algebra we obtain the following equations, with respect to the ADM basis:^b

$$C_{ijk}^{\mathbf{n}} = \left[\delta_i^r \delta_j^s \delta_k^t - \frac{1}{2} h^{rt} (h_{ik} \delta_j^s - h_{ij} \delta_k^s) \right] (K_{rs|t} - K_{rt|s}), \quad (5)$$

$$C_{ninj} = \frac{1}{2} \left(\delta_i^k \delta_j^l - \frac{1}{3} h_{ij} h^{kl} \right) \left(\mathcal{L}_{\vec{n}} K_{kl} + \frac{N_{|kl}}{N} + K K_{kl} + {}^{(3)}R_{kl} \right), \quad (6)$$

Owing to the fact that the Weyl tensor identically vanishes in three dimensions, the Lagrangian density in Eq. (1) reduces to

$$\mathfrak{L} = -N\sqrt{h} \left(2C_{ninj} C^{ninj} + C_{nijk} C^{nijk} \right), \quad (7)$$

^bLatin indices i, j, k, \dots refer to spacelike components in Σ_t ; the ADM basis is defined by $\vec{e}_{\mathbf{n}} := \vec{n} = (\partial_t - N^i \partial_i)/N$ and $\vec{e}_i = \partial_i$; a stroke ‘|’ denotes the covariant derivative onto Σ_t , and the symbol \mathbf{n} indicates a component along $\vec{e}_{\mathbf{n}}$.

where C_{nijk} and C_{ninj} are given by Eq. (5) and Eq. (6), respectively.

Now consider that the induced metric h_{ij} and the extrinsic curvature K_{ij} are *independent* variables — i.e., K_{ij} are introduced as auxiliary Ostrogradsky variables. In order to recover the definition $K_{ij} = -\frac{1}{2}\mathcal{L}_{\vec{n}}h_{ij}$ we must trade the original Lagrangian density for a constrained Lagrangian density

$$\underline{\mathcal{L}} = N^{-1}\mathfrak{L} + \lambda^{ij}(\mathcal{L}_{\vec{n}}h_{ij} + 2K_{ij}), \quad (8)$$

with Lagrange multipliers λ^{ij} as additional variables. Thus we must resort to Dirac's formalism for constrained systems.¹⁰ The generalized Ostrogradsky construction enables us to cast the action in Eq. (9) into canonical form,

$$S = \int_{\mathcal{M}} d^4x N [p^{ij} \mathcal{L}_{\vec{n}}h_{ij} + \mathcal{Q}^{ij} \mathcal{L}_{\vec{n}}K_{ij} - \mathfrak{H}_c(h, K, p, \mathcal{Q})], \quad (9)$$

with the conjugate momenta $p^{ij} = \lambda^{ij}$ and $\mathcal{Q}^{ij} = -2\sqrt{h}C^{ninj}$, and where the canonical Hamiltonian density is given by

$$\mathfrak{H}_c = -2p^{ij}K_{ij} + \sqrt{h}C_{nijk}C^{nijk} - \frac{\mathcal{Q}^{ij}\mathcal{Q}_{ij}}{2\sqrt{h}} - \mathcal{Q}^{ij}{}_{|ij} - \mathcal{Q}^{ij(3)}R_{ij} - K K_{ij}\mathcal{Q}^{ij}. \quad (10)$$

Dirac's constraint analysis yields, besides the usual super-Hamiltonian and super-momentum constraints, one first-class constraint that is the generator of conformal transformations. It reads explicitly

$$\chi = 2p + K_{kl}\mathcal{Q}^{kl} \approx 0. \quad (11)$$

Moreover we can get rid of the spurious canonical variables \mathcal{Q} and K ; only the traceless part of the corresponding tensors remain as relevant canonical variables. This is consistent with the fact that conformal gravity exhibits six degrees of freedom.

Performing a canonical transformation that we have defined in a previous work⁴ in order to disentangle terms stemming respectively from the pure R^2 and conformal variants of the general quadratic theory, we get

$$S = \int_{\mathcal{M}} d^4x [\Pi_{\mu}\dot{\mu} + \Pi_{+}\dot{\beta}_{+} + \Pi_{-}\dot{\beta}_{-} + \mathcal{Q}_{+}\dot{\mathcal{P}}_{+} + \mathcal{Q}_{-}\dot{\mathcal{P}}_{-} - N\mathcal{H}_c - \lambda_c\varphi_c], \quad (12)$$

where the first-class constraints $\mathcal{H}_c \approx 0$ (super-Hamiltonian) and $\varphi_c \approx 0$ are given respectively by the following expressions

$$\begin{aligned} \mathcal{H}_c = & -\frac{1}{\sqrt{6}} [\Pi_{+}\mathcal{P}_{+} + \Pi_{-}\mathcal{P}_{-} + 2\mathcal{Q}_{+}(\mathcal{P}_{+}^2 - \mathcal{P}_{-}^2) - 4\mathcal{P}_{+}\mathcal{P}_{-}\mathcal{Q}_{-}] \\ & -\frac{1}{2}e^{-3\mu}(\mathcal{Q}_{+}^2 + \mathcal{Q}_{-}^2), \end{aligned} \quad (13)$$

$$\varphi_c = \Pi_{\mu} - \mathcal{P}_{+}\mathcal{Q}_{+} - \mathcal{P}_{-}\mathcal{Q}_{-}. \quad (14)$$

A suitable gauge-fixing condition that eliminates variables μ and Π_μ , and the choice $N = e^\mu$ yield the final form of the canonical action

$$S = \int_{\mathcal{M}} d^4x \left[\Pi_+ \dot{\beta}_+ + \Pi_- \dot{\beta}_- + \mathcal{Q}_+ \dot{\mathcal{P}}_+ + \mathcal{Q}_- \dot{\mathcal{P}}_- - \mathcal{H}_c \right], \quad (15)$$

where the super-Hamiltonian is now given by the following expression

$$\mathcal{H}_c = \frac{1}{\sqrt{6}} [4\mathcal{P}_+ \mathcal{P}_- \mathcal{Q}_- - \Pi_+ \mathcal{P}_+ - \Pi_- \mathcal{P}_- - 2\mathcal{Q}_+ (\mathcal{P}_+^2 - \mathcal{P}_-^2)] - \frac{1}{2} (\mathcal{Q}_+^2 + \mathcal{Q}_-^2). \quad (16)$$

Varying the action as given by Eq. (15) with respect to the remaining canonical variables and their conjugate momenta, we obtain the canonical equations for the Bianchi type-I model in conformal gravity ^c

$$\dot{\beta}_\pm = -\frac{1}{\sqrt{6}} \mathcal{P}_\pm, \quad (17)$$

$$\dot{\Pi}_\pm = 0, \quad (18)$$

$$\dot{\mathcal{Q}}_+ = -\frac{1}{\sqrt{6}} (\Pi_+ + 4\mathcal{P}_- \mathcal{Q}_- - 4\mathcal{P}_+ \mathcal{Q}_+), \quad (19)$$

$$\dot{\mathcal{Q}}_- = -\frac{1}{\sqrt{6}} (\Pi_- + 4\mathcal{P}_- \mathcal{Q}_+ + 4\mathcal{P}_+ \mathcal{Q}_-), \quad (20)$$

$$\dot{\mathcal{P}}_+ = \frac{2}{\sqrt{6}} (\mathcal{P}_-^2 - \mathcal{P}_+^2) + \mathcal{Q}_+, \quad (21)$$

$$\dot{\mathcal{P}}_- = \frac{4}{\sqrt{6}} \mathcal{P}_+ \mathcal{P}_- + \mathcal{Q}_-. \quad (22)$$

Instead of the fourth-order Bach equations, we have now at our disposal the nice differential system, given by Eq. (17) to Eq. (22), which is more appropriate for applying singularity analysis methods (Painlevé test) in order to extract all the exact solutions. In contrast with these methods which probe the local analytical structure of the canonical system, we can also seek out exact solutions by performing a global involution algorithm on specific extra constraints chosen in accordance with local results from the analytic structure.

3.2 Global involution of extra constraints

The involution method consists in applying the Dirac–Bergmann consistency algorithm on our system, with the Poisson brackets defined with respect to the

^cTo enable a straightforward comparison with Boulware’s formalism we also have performed a canonical transformation that interchanges the coordinates and momenta \mathcal{P}_\pm and \mathcal{Q}_\pm .

canonical variables β_{\pm} , Π_{\pm} , Q_{\pm} , \mathcal{P}_{\pm} , and after suitable conditions have been imposed. Strictly speaking, the steps of the global involution algorithm are the following:

- Impose an appropriate extra constraint on the canonical variables;
- Require that constraint to be preserved when time evolution is considered. This gives rise to secondary constraints and possibly to the determination of the Lagrange multiplier associated with the extra constraint;
- Repeat the second step (involution) until no new information comes out.

Once the involution algorithm has been performed we can classify all the constraints into first class and second class and proceed further to the analysis of the particular system.

Among the set of constraints we have considered in our analysis two are more significant. The first expresses that the ratio of the variables \mathcal{P}_{\pm} is constant. Any solution to the canonical equations that satisfies that specific constraint is conformally equivalent to an Einstein space. Consistency of the extra constraint yields only one secondary constraint and both are second class. We eliminate the associated spurious degrees of freedom and reduce the canonical equations (19–22) to one binomial equation of Briot and Bouquet. The representations of the solution on the real axis are complicated expressions involving the Weierstrass elliptic function.⁶ As a particular case of that analysis we obtain the general axisymmetric solution. The second interesting constraint requires that the momenta Π_{\pm} be zero. In accordance with local results from the analytic structure, the general solution to the complete system with $\Pi_{\pm} = 0$ is easy to produce under analytic form, upon integrating linear ODE's of the Lamé type. It provides a specific example of a solution to the Bach equations that is not conformally related to an Einstein space.⁶

4 Analytic structure of Bianchi type-I cosmology

The existence of the above solutions, whether particular solutions of the general differential system or general solutions of specialized systems, tells nothing about the integrability or non-integrability of the complete system and gives no information whatsoever about the mere accessibility of an exact and closed-form analytic expression of its general solution. This is due to the fact that the global involution algorithm of the extra constraints, as operated above, is not related with integrability and may even prove to be non-exhaustive. We have tackled the integrability issue through an invariant investigation method

of intrinsic properties of the general solution.^d In particular, we have proved analytically that the system under consideration is not integrable: its general solution exhibits, in complex time, an infinite number of logarithmic transcendental essential movable singularities — i.e., an analytic structure not compatible with integrability in the practical sense; the quest for generic, exact and closed-form analytic expressions of the solution is hopeless. We stress that this result holds under spacetime transformations within the equivalence class of the Painlevé property.¹¹

5 Conformal relationship with Einstein spaces

The vanishing of the Bach tensor is a necessary condition for a Riemann space to be conformal to an Einstein space. It means that any solution of vacuum general relativity or any space conformal to an Einstein space are also a solutions of conformal gravity. The converse however is not true: there exist spaces not conformally related to Einstein spaces that satisfy the Bach equations.¹² A necessary and sufficient condition for a space to be conformal to an Einstein space is the existence of a function $\sigma(\mathbf{x})$, i.e., conformal factor, that satisfies the differential equations

$$L_{ab} = \nabla_a \sigma \nabla_b \sigma - \nabla_a \nabla_b \sigma - \frac{1}{2} g_{ab} g^{cd} \nabla_c \sigma \nabla_d \sigma - \frac{\tilde{R}}{24} e^{2\sigma} g_{ab} = 0, \quad (23)$$

where the tensor L_{ab} is defined by $L_{ab} := (Rg_{ab} - 6R_{ab})/12$. For the Bianchi type-I model it is not difficult to show that the above conditions in Eq. (23) imply precisely that the ratio of variables \mathcal{P}_{\pm} is constant (our first extra constraint in the algorithm). In that case the conformal factor can be uniquely determined as a function involving the Weierstrass elliptic function. Moreover it is possible to obtain the explicit form of the constant scalar curvature, \tilde{R} , of the conformal Einstein space. On the other hand, our solution obtained by imposing $\Pi_{\pm} = 0$ is not conformal to an Einstein space, for the ratio of variables \mathcal{P}_{\pm} is not constant in that case. We have thus confirmed explicitly Schmidt's conjecture of the existence of what he calls 'non-trivial' solutions to the Bach equations.¹³

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^dFor a circumstantial analysis, we refer the reader to our forthcoming paper.⁶

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